Feynman gauge on the lattice: new results and perspectives

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Abstract. We have recently introduced a new implementation of the Feynman gauge on the lattice, based on a minimizing functional that extends in a natural way the Landau-gauge case, while preserving all the properties of the continuum formulation. The only remaining difficulty with our approach is that, using the standard (compact) discretization, the gluon field is bounded, while its four-divergence satisfies a Gaussian distribution, i.e. it is unbounded. This can give rise to convergence problems when a numerical implementation is attempted. In order to overcome this problem, one can use different discretizations for the gluon field, or consider an SU(N_c) group with sufficiently large N_c. Here we discuss these two possible solutions.

INTRODUCTION

The behavior of Green's functions in the infrared limit of Yang-Mills theories should give us some insights into the low-energy properties of these theories. Since these functions depend on the gauge condition, considering different gauges could help us gain a better understanding of the (non-perturbative) low-energy hallmarks of QCD, such as color confinement. In the last 20 years, several groups have used lattice simulations to study propagators and vertices of Yang-Mills theories in Landau gauge [1], Coulomb gauge [2, 3, 4], λ-gauge (a gauge that interpolates between Landau and Coulomb) [5, 6] and maximally Abelian gauge [7, 8].

On the other hand, until recently, the numerical gauge fixing for the linear covariant gauge — which is a generalization of Landau gauge — was not satisfactory [9, 10, 11, 12, 13, 14, 15, 16]. In Ref. [17] we have introduced a new implementation of the linear covariant gauge on the lattice that solves most problems encountered in earlier implementations (see [17, 18] for a short review of early works). As explained in the abstract, the only problem still affecting our method, as well as any formulation of the linear covariant gauge on the lattice, is due to the fact that the gluon field $A_\mu^a(x)$ is *bounded* in the usual compact formulation of lattice Yang-Mills theories. On the contrary, the functions $\Lambda^b(x)$ satisfy a Gaussian distribution, i.e. they are *unbounded*. Thus, one has to deal with convergence problems [19] when numerically fixing the gauge

condition

$$\partial_\mu A^b_\mu(x) = \Lambda^b(x) \ . \tag{1}$$

Since the real-valued functions $\Lambda^b(x)$ are generated using a Gaussian distribution with width $\sqrt{\xi}$, it is clear that this problem becomes more severe when ξ is larger and/or when the lattice volume is larger. Here we discuss two possible solutions for this problem, namely we consider different discretizations for the gluon field or a gauge group SU(N_c) with sufficiently large N_c.

LINEAR COVARIANT GAUGE ON THE LATTICE

We want to impose the gauge condition (1) on the lattice. Landau gauge, which corresponds to the case $\Lambda^b(x) = 0$, is obtained on the lattice by minimizing the functional

$$\mathscr{E}_{LG}[U^g] = -\text{Tr}\sum_{\mu,x} g(x)U_\mu(x)g^\dagger(x+e_\mu) \ . \tag{2}$$

Here $U_\mu(x)$ are link variables and $g(x)$ are site variables, both belonging to the SU(N_c) group. The sum is taken over all lattice sites x and directions μ. Also, Tr indicates trace in color space. For the linear covariant gauge we can look for a minimizing functional of the type $\mathscr{E}_{LCG}[U^g, g, \Lambda]$. If one recalls that solving the system of equations $B\phi = c$, where B is a matrix and ϕ and c are vectors, is equivalent to minimizing the quadratic form $\frac{1}{2}\phi B\phi - \phi c$, then it is obvious that in our case we should look for a minimizing functional of the type [17]

$$\mathscr{E}_{LCG}[U^g, g, \Lambda] \sim \mathscr{E}_{LG}[U^g] - g\Lambda \ . \tag{3}$$

Indeed, the lattice linear covariant gauge condition can be obtained by minimizing[1]

$$\mathscr{E}_{LCG}[U^g, g, \Lambda] = \mathscr{E}_{LG}[U^g] + \Re\, Tr\sum_x i g(x)\Lambda(x) \ , \tag{4}$$

where $\mathscr{E}_{LG}[U^g]$ is defined above in Eq. (2) and $\Re$ indicates real part. This can be checked by considering a one-parameter subgroup $g(x,\tau) = \exp\left[i\tau\gamma^b(x)\lambda^b\right]$. Here we indicate with λ^b a basis for the SU(N_c) Lie algebra and with $\gamma^b(x)$ any real-valued functions. Indeed, the stationarity condition implies the lattice linear covariant gauge condition[2]

$$\sum_\mu A^b_\mu(x) - A^b_\mu(x-e_\mu) = \Lambda^b(x) \ . \tag{5}$$

Also, the second variation (with respect to the parameter τ) of the term $i g(x)\Lambda(x)$ is purely imaginary and it does not contribute to the Faddeev-Popov matrix $\mathscr{M}$, i.e. $\mathscr{M}$

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[1] One should stress that, in the minimization process, the link variables $U_\mu(x)$ get gauge-transformed to $g(x)U_\mu(x)g^\dagger(x+e_\mu)$, while the $\Lambda^b(x)$ functions do not get modified.

[2] Note that periodic boundary conditions yields $\sum_x \Lambda^b(x) = 0$. This equality has to be enforced explicitly, within machine precision, when the functions $\Lambda^b(x)$ are generated.

TABLE 1. Smallest value of β for which the numerical gauge-fixing algorithm showed convergence. Results are reported for the three different discretizations and for five different values of the gauge parameter ξ.

ξ	stand.	angle	stereog.
0.01	2.2	2.2	2.2
0.05	2.2	2.2	2.2
0.1	2.2	2.2	2.2
0.5	2.8	2.6	2.5
1.0	—	3.0	2.5

is a discretized version of the usual Faddeev-Popov operator $-\partial \cdot D$. Let us note that having a minimizing functional for the linear covariant gauge implies that the Faddeev-Popov operator $\mathscr{M}$ is positive-definite and that the set of its local minima defines the first Gribov region Ω.[3]

It is interesting to note that one can interpret the Landau-gauge functional $\mathscr{E}_{LG}[U^g]$ as a spin-glass Hamiltonian [21] for the spin variables $g(x)$ with a random interaction given by $U_\mu(x)$. Then, our new functional corresponds to the same spin-glass Hamiltonian when a random external magnetic field $\Lambda(x)$ is applied.

Note also that the functional $\mathscr{E}_{LCG}[U^g, g, \Lambda]$ is linear in the gauge transformation $\{g(x)\}$. Thus, one can easily extend to the linear covariant gauge the gauge-fixing algorithms usually employed in the Landau case [22, 23, 24]. We refer the reader to References [17, 18] for tests of convergence of the numerical gauge fixing. There we have also checked that the quantity $D_l(p^2)p^2$, where $D_l(p^2)$ is the longitudinal gluon propagator, is approximately constant for all cases considered, as predicted by Slavnov-Taylor identities. This verification failed in previous formulations of the lattice linear covariant gauge [13, 16].

DISCRETIZATION EFFECTS

As explained above, the standard discretization of the gluon field $A_\mu^a(x)$ is bounded. Since the functions $\Lambda^b(x)$ are generated using a Gaussian distribution, it is clear that Eq. (5) cannot be satisfied if $\Lambda^b(x)$ is too large. A possible solution to this problem is to use different discretizations of the gluon field. We did some tests in the SU(2) case using the angle (or logarithmic) projection [25] and the stereographic projection [26] (for a slightly different implementation of the stereographic projection see also [27]). Note that, in the latter case, the gluon field is unbounded even for a finite lattice spacing a.

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[3] This region has been studied analytically in [20], for a small value of ξ, but a similar numerical study is still lacking.

TABLE 2. Values of the lattice coupling β considered for the gauge groups SU(2), SU(3) and SU(4) with a gauge parameter $\xi = 1$.

N_c	β_1	β_2	β_3	β_4
2	3.0	2.485	2.295	2.44
3	6.75	6.67	6.07	5.99
4	12.0	12.59	11.43	10.97

In particular, considering the standard discretization, the angle projection and the stereographic projection for the lattice volume $V = 8^4$, gauge parameter $\xi = 0.01, 0.05, 0.1, 0.5, 1.0$ and lattice coupling $\beta = 2.2, 2.3, \ldots, 2.9, 3.0$ we checked (using for the numerical gauge fixing the so-called Cornell method [22, 23, 24]) in which cases we were able to fix the covariant gauge condition effectively. Our results, reported in Table 1, clearly show that the angle projection is already an improvement compared to the standard discretization and that the best convergence is obtained when using the stereographic projection.

CONTINUUM LIMIT

Note [16] that the continuum relation

$$\partial_\mu A_\mu^b(x) = \Lambda^b(x) \tag{6}$$

can be made dimensionless — working in a generic d-dimensional space — by multiplying both sides by $a^2 g_0$. Since $\beta = 2N_c/(a^{4-d}g_0^2)$ [in the SU(N_c) case], we have that the lattice quantity

$$\frac{\beta/(2N_c)}{2\xi} \sum_{x,b} \left[a^2 g_0 \Lambda^b(x) \right]^2 = \frac{1}{2\sigma^2} \sum_{x,b} \left[a^2 g_0 \Lambda^b(x) \right]^2 \tag{7}$$

becomes

$$\frac{1}{2\xi} \frac{1}{a^{4-d}g_0^2} \int \frac{d^d x}{a^d} \sum_b \left[a^2 g_0 \Lambda^b(x) \right]^2 = \frac{1}{2\xi} \int d^d x \sum_b \left[\Lambda^b(x) \right]^2 \tag{8}$$

in the formal continuum limit. Thus, if we consider a gauge parameter ξ in the continuum, the lattice quantity $a^2 g_0 \Lambda^b(x)$ is generated from a Gaussian distribution with width $\sigma = \sqrt{2N_c \xi / \beta}$, instead of a width $\sqrt{\xi}$.

Note that $\sigma = \sqrt{\xi}$ if $\beta = 2N_c$ and that for $\beta < 2N_c$ the lattice width σ is larger than the continuum width $\sqrt{\xi}$, making the convergence problem discussed above more severe. Thus, in the SU(2) case, one has $\sigma = \sqrt{\xi}$ only for $\beta = 4$, corresponding to a lattice spacing $a \approx 0.001$ fm. On the contrary, in the SU(3) case, one has $\sigma = \sqrt{\xi}$ for $\beta = 6$, corresponding to $a = 0.102$ fm. Also, for a fixed t'Hooft coupling $g_0^2 N_c$, we have

TABLE 3. Values of β for which the numerical gauge-fixing algorithm showed convergence. Results are reported for three different gauge groups and four different lattice volumes. In all cases the gauge parameter ξ was 1 (Feynman gauge). *[In these two cases only a few configurations have been considered and more tests are needed.]

	8^4	16^4	24^4	32^4
SU(2)	β_1, β_2	—	—	—
SU(3)	all	β_1, β_2	β_1, β_2	β_1, β_2 *
SU(4)	all	all	all	$\beta_1, \beta_2, \beta_3$

$\beta \propto N_c^2$ and $\sigma \propto \sqrt{1/N_c}$. This suggests that simulations for the linear covariant gauge are probably easier in the SU(N_c) case for large N_c.

In order to test this hypothesis we simulated the SU(2), SU(3) and SU(4) cases for a gauge parameter $\xi = 1$ and lattice volumes $V = 8^4, 16^4, 24^4, 32^4$ for the values of β reported in Table 2. They correspond, respectively, to a t'Hooft coupling $g_0^2 N_c = 8/3$ (β_1), to a plaquette average value of about 0.65 (β_2) and of about 0.6 (β_3) and to a string tension (in lattice units) of about $a^2 \sigma = 0.044$ (β_4), giving $a \approx 0.09$ fm. In Table 3 we present, for each pair (SU(N_c), V), the values of the lattice coupling β for which the gauge-fixing algorithm showed a numerical convergence.[4] One clearly sees that the situation improves when the number of colors N_c is larger.

CONCLUSIONS

We have recently introduced a minimizing functional for the linear covariant gauge which is a simple generalization of the Landau-gauge functional. The new approach solves most problems encountered in earlier implementations and ensures a good quality for the gauge fixing. Here we have shown that, by using different discretizations for the gluon field (such as the angle projection) and by considering a gauge group SU(N_c) with N_c sufficiently large — i.e. $N_c = 4$ or maybe even $N_c = 3$ — one should be able to do simulations for $\xi = 1$ (Feynman gauge) and for large lattice volumes (in physical units). Let us note that a numerical study of the infrared behavior of propagators and vertices at $\xi \neq 0$ could provide important inputs for analytic studies based on Dyson-Schwinger equations [28, 29]. Moreover, it has been proven [30, 31, 32] that the background-field Feynman gauge is equivalent (to all orders in perturbation theory) to the pinch technique [32, 33]. Thus, numerical studies using the Feynman gauge, which corresponds to the value $\xi = 1$, could allow a nonperturbative evaluation of the gauge-invariant off-shell Green functions of the pinch technique [34].

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[4] For these tests we used the standard overrelaxation algorithm [22, 23, 24].

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